PSTAT 100: Lecture 17

Classification

Ethan P. Marzban

Department of Statistics and Applied Probability; UCSB

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RMS Titanic

- The RMS Titanic was an ocean liner that set sail from Southampton (UK) to New York (US) on April 10, 1912.

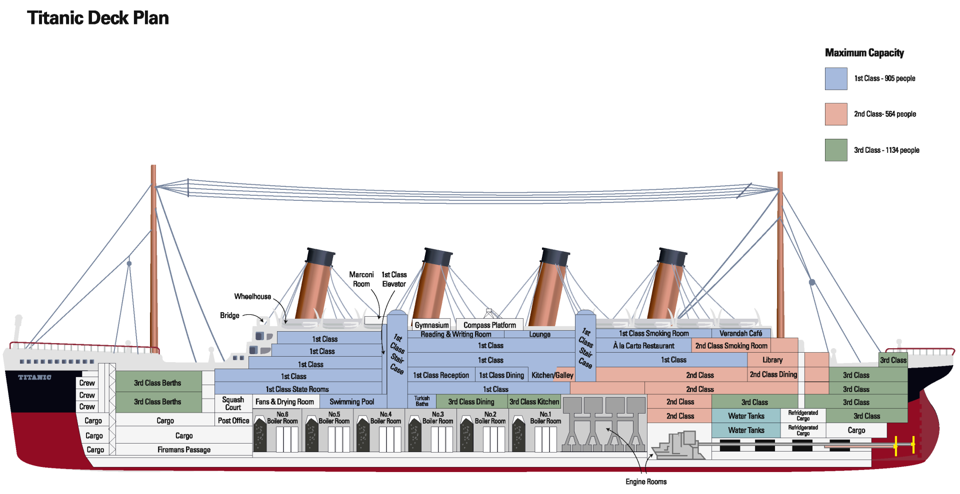


Image Source: https://images.liverpoolmuseums.org.uk/2020-01/titanic-deck-plan-for-titanic-resource-pack-pdf.pdf

- 5 days into its journey, on April 15, 1912, the ship collided with an iceberg and sank.
  - → Tragically, the number of lifeboats was far fewer than the total number of passengers, and as a result not everyone survived.

- A passenger/crew manifest still exists, which includes survival statuses.

RMS Titanic

```r
titanic <- read.csv("data/titatnic.csv")
titanic %>% head(3) %>% pander()
```

Table continues below

PassengerId	Survived	Pclass	Name	Sex	Age
1	0	3	Braund, Mr. Owen Harris	male	22
2	1	1	Cumings, Mrs. John Bradley (Florence Briggs Thayer)	female	38
3	1	3	Heikkinen, Miss. Laina	female	26

SibSp	Parch	Ticket	Fare	Cabin	Embarked
1	0	A/5 21171	7.25		S
1	0	PC 17599	71.28	C85	C
0	0	STON/O2. 3101282	7.925		S

RMS Titanic

- **Question:** given a passenger's information (e.g. sex, class, etc.), can we predict whether or not they would have survived the crash?
- Firstly, based on *domain knowledge* available to us, we believe there to be a relationship between survival rates and demographics.
  - → For example, it is known that women and children were allowed to board lifeboats before adult men; hence, it's plausible to surmise that women and children had higher survival rates than men.
  - → Additionally, lifeboats were located on the main deck of the ship; so, perhaps those staying on higher decks had greater chances of survival than those staying on lower decks.

RMS Titanic

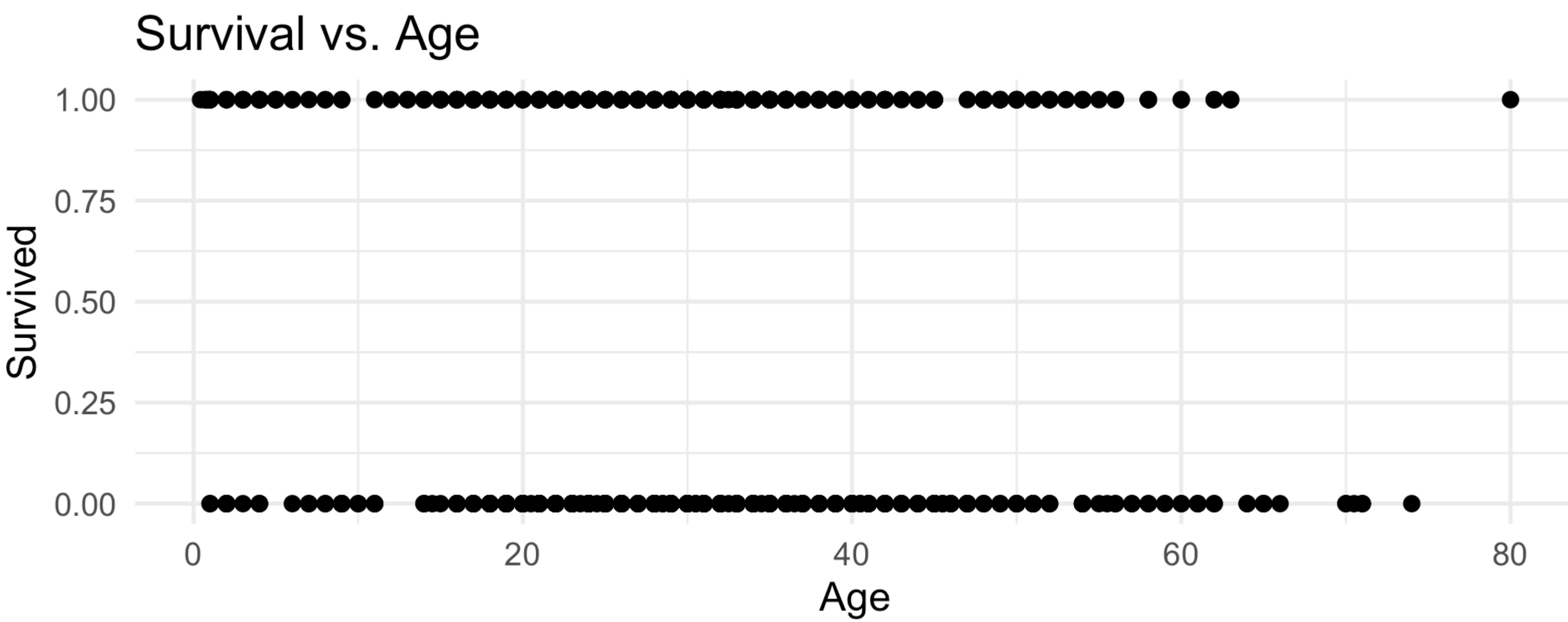
First Model

- To make things more explicit, let's suppose we wish to predict survival based solely on a passenger's age.
- This lends itself nicely to a model, with:
  - → **Response:** survival status (either `1` for `survived`, or `0` for `died`)
  - → **Predictor:** age (numerical, continuous)
- Now, note that our response is *categorical*. Hence, our model is a *classification* model, as opposed to a *regression* one.
- The (parametric) modeling approach is still the same:
  1. Propose a model
  2. Estimate parameters
  3. Assess model fit

RMS Titanic

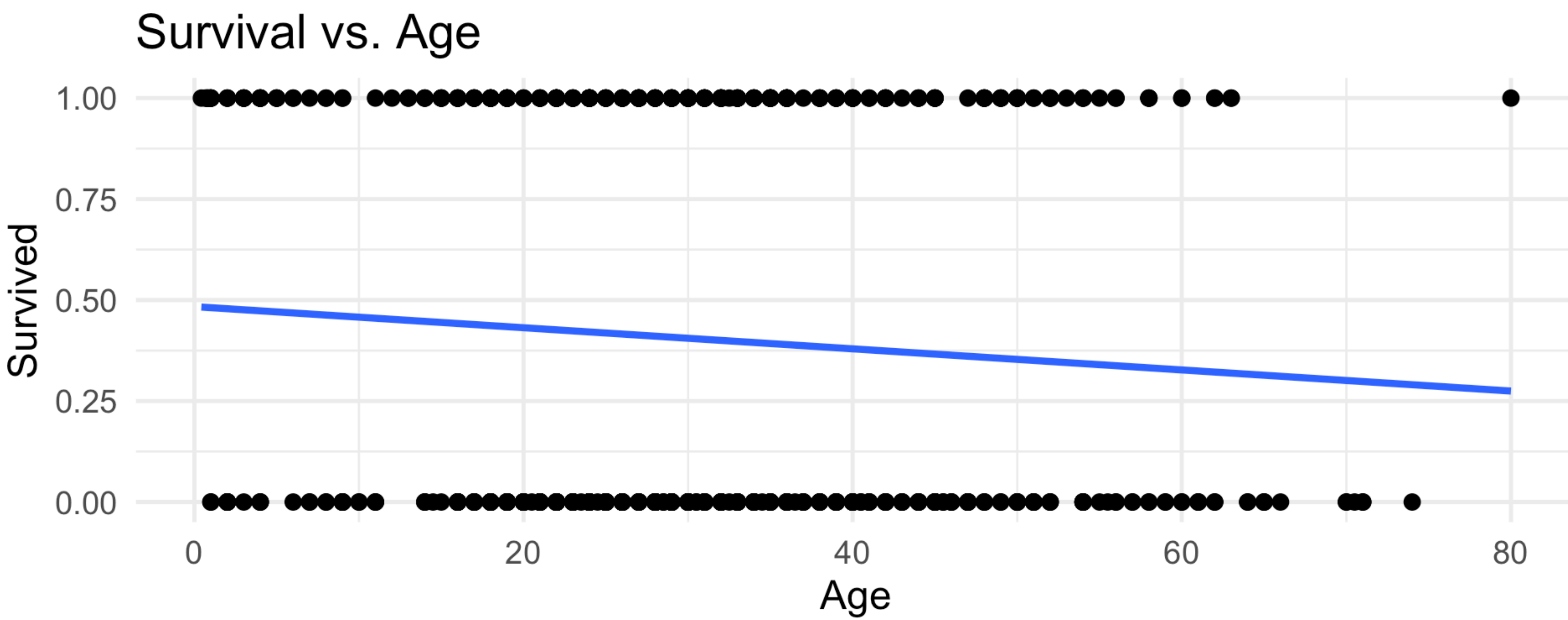
First Model

- We just have to be a bit more creative about our model proposition.
- Let's see what happens if we try to fit a "linear" model: $\mathbf{y_i} = \beta_0 + \beta_1 \mathbf{x_i} + \varepsilon_i$



RMS Titanic

First Model



- But what does this line *mean*?
  → The problem is in our proposed model.

RMS Titanic

First Model

$$\mathbf{y_i} = \beta_0 + \beta_1 \mathbf{x_i} + \varepsilon_i$$

- For any *i*, $\mathbf{y_i}$ will either be zero or one.
- But, for any *i*, $\mathbf{x_i}$ will be a positive number, not necessarily constrained to be either 0 or 1.
- So, this model makes no sense; how can something that is categorical equal something that is numerical?
- There are a couple of different resolutions - what we discuss in PSTAT 100 is just one possible approach.

RMS Titanic

Second Model

- **First Idea:** rethink the way we incorporate randomness (error) into our model.
  - → Let's define the random variable Y_i to be the survival status of the i^{th} (randomly selected) passenger. Then $Y_i \sim \text{Bern}(\pi_i)$, where π_i denotes the probability that the i^{th} (randomly selected) passenger survives.
- **Second Idea:** instead of modeling Y_i directly, model the *survival probabilities*, π_i.
  - → After all, the probability of surviving is likely related to age.
- But, $\pi_i = \beta_0 + \beta_1 \mathbf{x_i}$ is *still* not a valid model, since π_i is constrained to be between 0 and 1, whereas $(\beta_0 + \beta_1 \mathbf{x_i})$ is unconstrained.

RMS Titanic

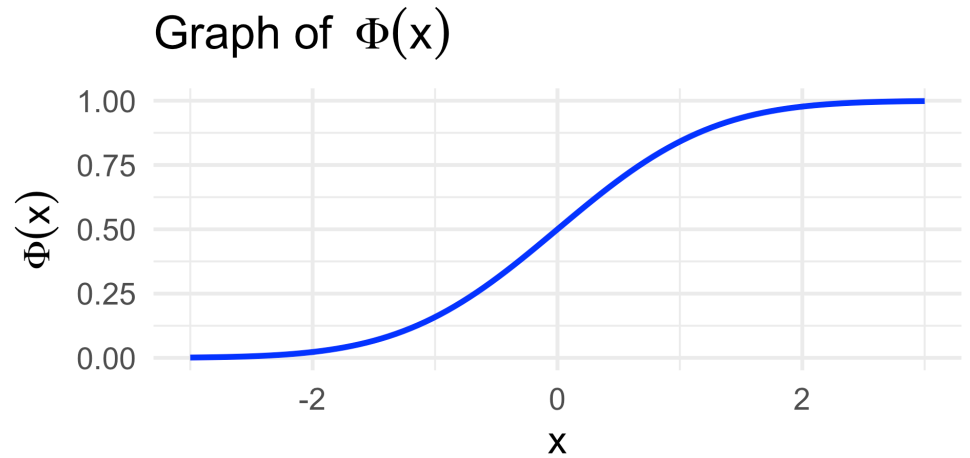
Second Model

- **Third Idea:** apply a *transformation* to $\beta_0 + \beta_1 \mathbf{x_i}$.
- Specifically, if we can find a function *g* that maps from the real line to the unit interval, then a valid model would be $\pi_i = g(\beta_0 + \beta_1 \mathbf{x_i})$.
- What class of (probabilistic) functions map from the real line to the unit interval?
  - → CDF's!
- Indeed, we can pick *any* CDF to be our transformation *g*. There are two popular choices, giving rise to two different models:
  - → Standard Normal CDF, leading to **probit** models
  - → Logistic Distribution CDF, leading to **logit** models

Probit vs. Logit Models

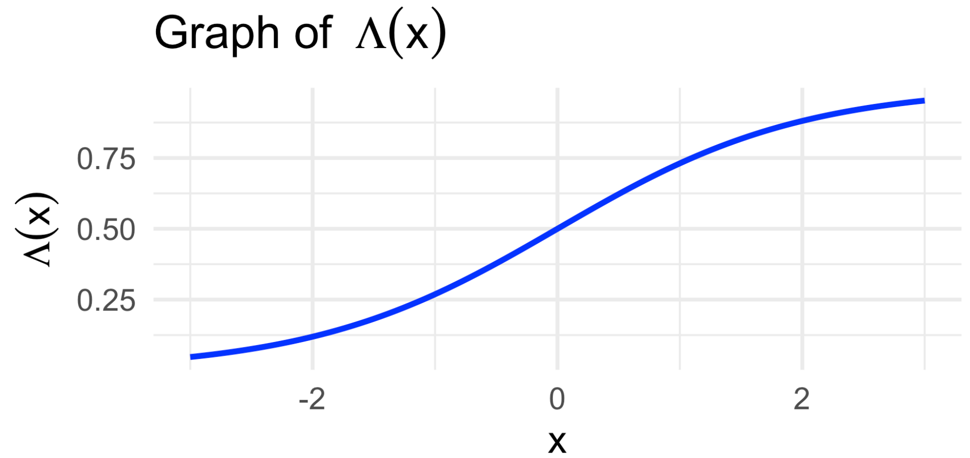
Probit Model: $\pi_i = \Phi(\beta_0 + \beta_1 \mathbf{x_i})$

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, \mathrm{d}z$$



Logit Model: $\pi_i = \Lambda(\beta_0 + \beta_1 \mathbf{x_i})$

$$\Lambda(x) := \frac{1}{1 + e^{-x}}$$



Logistic Regression

- As an example, let's return to our *Titanic* example where $\pi_{\texttt{i}}$ represents the probability that the i^{th} passenger survived, and $\mathbf{x_i}$ denotes the i^{th} passenger's age.
- A logistic regression model posits

$$\pi_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 x_i)}}$$

- Equivalently,

$$\ln\left(\frac{\pi_i}{1 - \pi_i}\right) = \beta_0 + \beta_1 x_i$$

→ **Aside:** we call the function $g(t) = \ln(t / (1 - t))$ the **logit function**.

Logistic Regression

Model Assumptions

Logistic Regression

- The second formulation of our model makes it a bit easier to interpret the coefficients:
  - → *Ceterus paribus* (holding all else constant), a one-unit increase in x_i is modeled to be associated with a β_1 -unit increase in the **log-odds** of π_i.
  - → β_0 represents the log-odds of survival of a unit with a predictor value of zero.
- In `R`, we fit a logistic regression using the `glm()` function.
  - → This is because logistic regression is a special type of what is known as a **Generalized Linear Model** (GLM), which is discussed further in PSTAT 127.

Logistic Regression

Titatnic Dataset

```r
glm(Survived ~ Age, data = titanic, family = "binomial") %>% summary
```

```

Call:
glm(formula = Survived ~ Age, family = "binomial", data = titanic)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)  
(Intercept) -0.05672    0.17358  -0.327   0.7438  
Age         -0.01096    0.00533  -2.057   0.0397 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)
```

$$\ln\left(\frac{\widehat{\pi}_i}{1-\widehat{\pi}_i}\right) = -0.05672 - 0.01096x_i$$

Logistic Regression

Titatnic Dataset

- So, as expected, a one-unit increase in `age` corresponds to a *decrease* in the log-odds of survival.
  - → Again, this is "expected" because we know children were allowed to board lifeboats before adults.
- By the way, can anyone tell me why we use `family = "binomial"` in our call to `glm()`?
  - → Specifically, what is "binomial" about our logistic regression model? (Hint: go back to the beginning of how we constructed our model!)
- **Example Question:** Karla was around 24 years old. What is the probability that she would have survived the crash of the Titanic?

Logistic Regression

Titatnic Dataset

```r
glm_age <- glm(Survived ~ Age, data = titanic, family = "binomial")
(p1 <- predict(glm_age, newdata = data.frame(Age = 24)))
```

```
         1 
-0.3198465 
```

Caution

`predict.glm()` will give you the predicted *log-odds* - to find the true predicted *survival probability*, you need to invert.

```r
1 / (1 + exp(-p1))
```

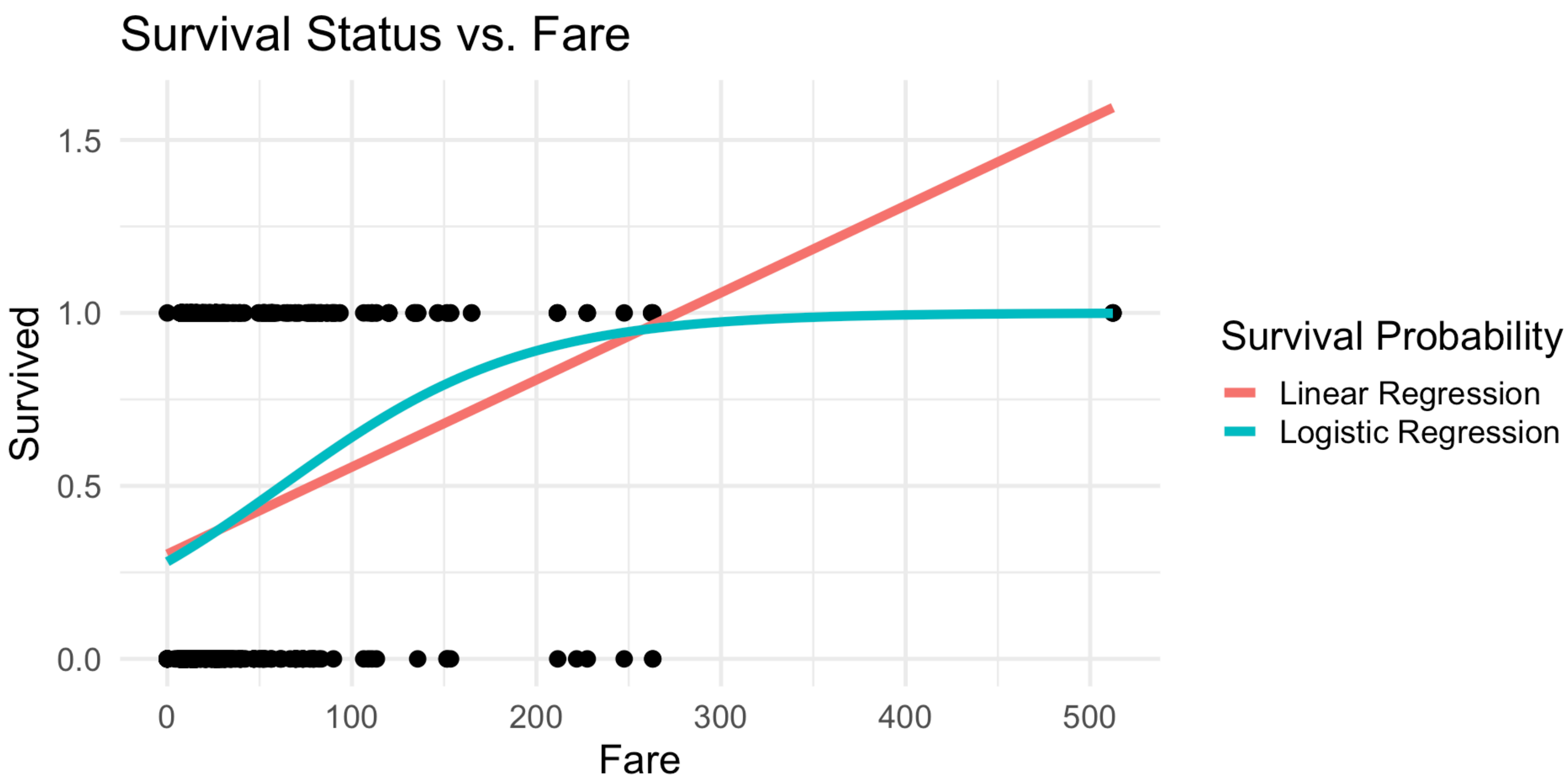
```
        1 
0.4207132 
```

- So, based on our model, Karla has an approximately 42.1% chance of having survived the crash of the Titanic.

Logistic Regression

Titatnic Dataset

▶ Code



- Does this make sense, based on our background knowledge?

Multiple Logistic Regression

- Of course, we can construct a logistic regression with *multiple* predictors:

$$\pi_i = \Lambda\left(\beta_0 + \sum_{j=1}^{p} \beta_j x_{ij}\right) = \frac{1}{1 - e^{-\left(\beta_0 + \sum_{j=1}^{p} \beta_j x_{ij}\right)}}$$

$$\mathrm{logit}(\pi_i) = \beta_0 + \sum_{j=1}^{p} \beta_j x_{ij}$$

- Estimating the parameters ends up being a task and a half; indeed, there do not exist closed-form solutions for the optimal estimates.

  → Instead, most computer programs utilize recursive algorithms to perform the model fits.

Your Turn!

Your Turn!

Adebimpe has found that a good predictor of whether an email is spam or not is the number of times the word "promotion" appears in its body. To that end, she has fit a logistic regression model to model an email's spam/ham status as it relates to the number of times the word "promotion" appears. The resulting regression table is displayed below:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.68748    0.04360  15.768  < 2e-16 ***
num_prom     0.10258    0.01844   5.564  1.2e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

a. What is the predicted probability that an email containing the word "promotion" 3 times is spam?

b. Provide an interpretation for the value `0.01844` in the context of this problem.

03:00

Classification

- Now, logistic regression gets us estimated survival *probabilities*.
- It does not, however, give us survival *statuses* - to get those, we need to build a **classifier**.
  - → For example, a few slides ago we said that 24-year-old Karla had a 42.1% chance of surviving the crash of the *Titanic*.
  - → But, if she were an actual passenger on the *Titanic* she would have either survived or not.
- In binary classification (i.e. where our original response takes only two values, `survived` or `not`), our classifier typically takes the form: assign y_i a value of `survived` if the survival probability is above some threshold *c*, and assign y_i a value of `did not survive` if the survival probability falls below the threshold.

Titanic Classifier

- To start, let's explore the following classifier: $\{Y_i = 1\}$ if and only if the predicted survival probability was above 50%.
  - → Let's also stick with our model that models survival probabilities in terms of only `Fare`.

```
glm_fare <- glm(Survived ~ Fare, data = titanic, family = "binomial"
probs <- glm_fare$fitted.values
titanic$PassengerId[which(probs > 0.5)] %>% head(15)
```

```
[1]   2  28  32  35  53  55  62  63  73  89  98 103 119 121 125
```

- Can anyone tell me, in words, what these represent?

```
sum(titanic[which(probs > 0.5),]$Survived) / length(which(probs > 0.
```

```
[1] 0.6833333
```

- What does this represent?

Confusion Matrices

True Status

		Positive	Negative
Predicted Status (from Model)	**Positive**	TP (True Positive)	FP (False Positive)
	Negative	FN (False Negative)	TN (True Negative)

Sensitivity

$$\text{TPR} = \frac{\text{TP}}{\text{TP} + \text{FN}}$$

$$\text{FNR} = \frac{\text{FN}}{\text{TP} + \text{FN}}$$

Specificity

$$\text{TNR} = \frac{\text{TN}}{\text{FP} + \text{TN}}$$

$$\text{FPR} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

Confusion Matrices

- For example, in the context of the *Titanic* dataset:
  - → The count of **true positives** is the number of passengers correctly classified as having survived
  - → The count of **false negatives** is the number of passengers incorrectly classified as having died
- The **True Positive Rate** (aka **sensitivity**) is the proportion of passengers who actually survived that were correctly classified as having survived.
- The **False Positive Rate** (aka one minus the **specificity**) is the proportion of passengers who actually died that were incorrectly classified as having survived.

Confusion Matrices

Titanic Example

Classifier: $\{Y_i = 1\} \iff \{\widehat{\pi}_i > 0.5\}$

```
tp <- ((titanic$Survived == 1) * (fitted.values(glm_fare) > 0.5)) %>%
fp <- ((titanic$Survived == 0) * (fitted.values(glm_fare) > 0.5)) %>%

fn <- ((titanic$Survived == 1) * (fitted.values(glm_fare) < 0.5)) %>%
tn <- ((titanic$Survived == 0) * (fitted.values(glm_fare) < 0.5)) %>%
```

	truth_+	truth_-
class_+	82	38
class_-	260	511

```
TPR: 0.2397661

FPR: 0.06921676
```

Confusion Matrices

Titanic Example

Classifier: $\{Y_i = 1\} \iff \{\widehat{\pi}_i > 0.9\}$

```
tp <- ((titanic$Survived == 1) * (fitted.values(glm_fare) > 0.9)) %>%
fp <- ((titanic$Survived == 0) * (fitted.values(glm_fare) > 0.9)) %>%

fn <- ((titanic$Survived == 1) * (fitted.values(glm_fare) < 0.9)) %>%
tn <- ((titanic$Survived == 0) * (fitted.values(glm_fare) < 0.9)) %>%
```

	truth_+	truth_-
class_+	14	6
class_-	328	543

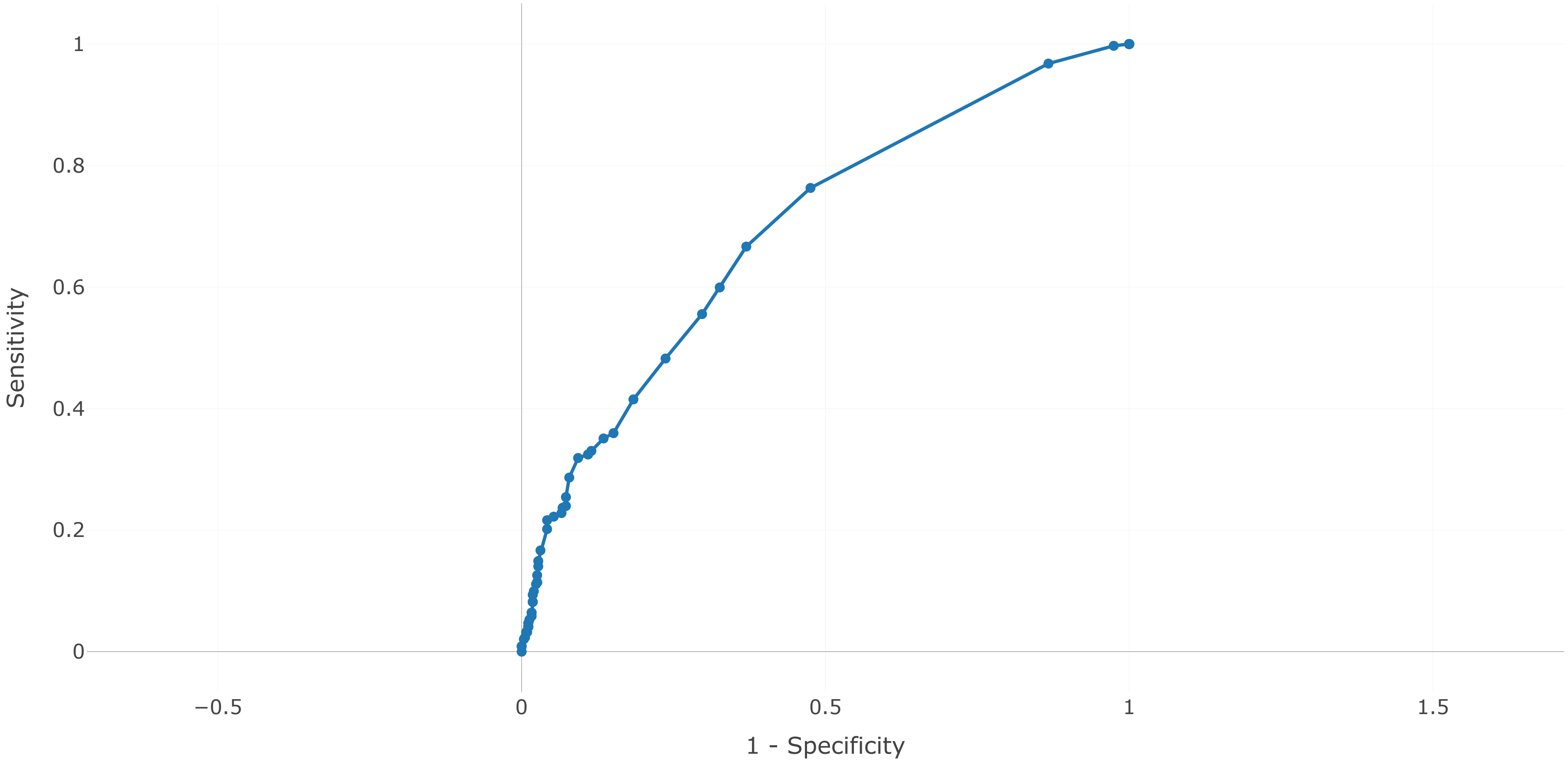
```
TPR: 0.04093567

 FPR: 0.01092896
```

Performance of a Classifier

ROC Curves

- So, we can see that our TPR and TNR will change depending on the cutoff value we select for our classifier.
- This gives us the idea to perhaps use quantities like TPR and TNR to compare across different cutoff values.
- Rather than trying to compare confusion matrices, it's a much nicer idea to try and compare plots.
- One such plot is called a **Receiver Operating Characteristic (ROC) Curve**, which plots the sensitivity (on the vertical axis) against (1 - specificity) (on the horizontal axis)

ROC Curves

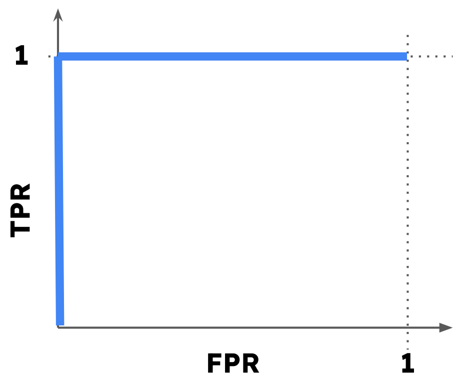


- We pick the cutoff to be that which makes the ROC curve as close to the point (0, 1) as possible.
  → This indicates we should use a cutoff of around 33%

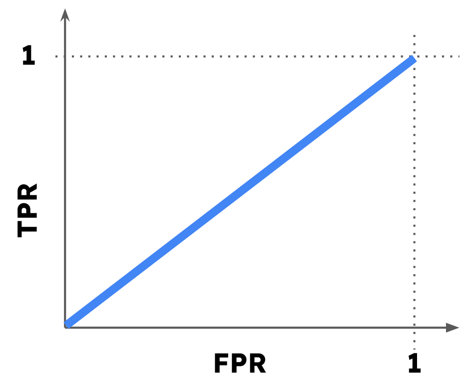
Performance of a Classifier

ROC Curves

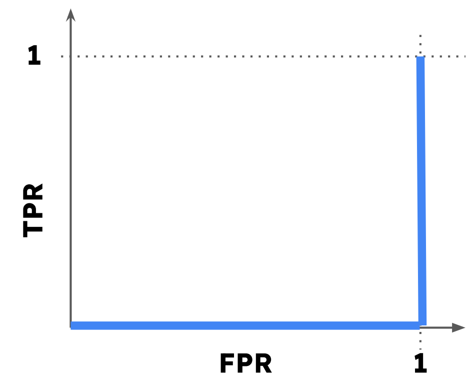
- Allow me to elaborate a bit more on this last point.
- The vertical axis of a ROC curve effectively represents the probability of a good thing; ideally, we'd like a classifier that has a 100% TPR!
- Simultaneously, an ideal classifier would also have a 0% FPR (which is precisely what is plotted on the horizontal axis of an ROC curve).



Perfect Classification;
Ideal



Random Classification;
Benchmark



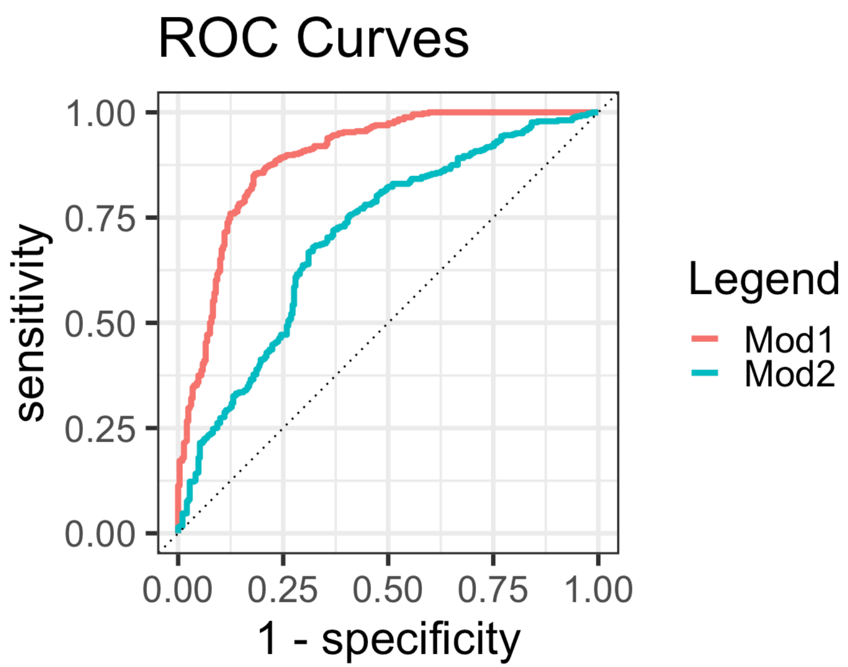
Perfect Misclassification;
Not Ideal

Performance of a Classifier

ROC Curves

- ROC curves can also be used to compare across *models* as well.

- **Model 1:** Using `Fare`, `Age`, `Sex`, and `Cabin` as predictors
- **Model 2:** Using `Fare` and `Age` as predictors



- The ROC curve for model 1 is farther from the diagonal than model 2, indicating that it is the better choice.

⏭ Next Time

- In lab today, you'll explore classification a bit further.
  - → Specifically, you'll work through fitting a few logistic models and building a few classifiers based on a non-simulated dataset pertaining to dates (the fruit)
- There will be no new material tomorrow; instead, we'll review for ICA 02.
  - → If you haven't already, please read through the information document I posted on the website pertaining to ICA 02.
  - → As a reminder, all material (up to and including today's lecture **and lab**) is potentially fair game for the ICA, though there will be a considerable emphasis on material from *after* ICA 01.
- Also, recall you'll be getting early-access to Lab08 solutions by going to Section today!